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OPTIMAL FILTERING OF LINEAR SYSTEM DRIVEN BY FRACTIONAL BROWNIAN MOTION

MASNITA MISIRAN[1], CHANGZI WU[2], ZUDI LU[3], AND K. L. TEO[1]

[1]Department of Mathematics and Statistics, Curtin University of Technology, Perth, Australia

[2]Department of Mathematics, Chongqing Normal University, Chongqing, China

[3]School of Mathematical Sciences, The University of Adelaide, Adelaide, Australia

ABSTRACT. In this paper, we consider a continuous time filtering of a multi-dimensional Langevin stochastic differential system driven by a fractional Brownian motion process. It is shown that this filtering problem is equivalent to an optimal control problem involving convolutional integrals in its dynamical system. Then, a novel approximation scheme is developed and applied to this optimal control problem. It yields a sequence of standard optimal control problems. The convergence of the approximate standard optimal control problem to the optimal control problem involving convolutional integrals in its system dynamics is established. Two numerical examples are solved by using the method proposed. The results obtained clearly demonstrate its efficiency and effectiveness.

Keywords: linear filtering, fractional Brownian motion, optimal control, convolutional integrals, approximation scheme, approximate optimal control computation

AMS (MOS) Subject Classification. 41A50

1. INTRODUCTION

Fractional Brownian motion (FBM) has been received with an increasing interest in many research fields in the past several decades, due to the early works of Mandelbrot and his collaborators. See, for examples, [23, 24, 25]. FBM allows a model to take into account the long-memory dependency, a notable property which is absent in the standard Brownian motion (BM). Since then, FBM, as a governing noise, has been applied to a variety of fields, from hydrology [24], network and telecommunication traffics [1, 30], to economics and finance [9, 28]. In this paper, we are concerned with a continuous time filtering of a multi-dimensional linear system driven by FBM in control theory.

The study of filtering has been around for several decades. It goes back as early as 1960, when Kalman [17] dealt with a problem posed by Gauss on the estimation of the satellite orbits. Later in 1961, Kalman and Bucy [18] studied the filtering problem involving linear continuous-time processes. It has been used in various areas

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arise in physical sciences, engineering, economics and social sciences. Its aim is to extract the best information on the state process based on the measured data. For further details, see, for example, [2, 4, 8, 14]. However, most of the filtering problems in the literature are concerned with noises characterized by standard BMs. There are only few results on the filtering problems which are given by FBM processes. In [11, 19, 20], only one dimensional differential equation is considered. In [3], the study is extended to a multi-dimensional case where both the state and observation are governed by respective linear stochastic differential equations which are driven by FBM processes. Our study is based on the fundamental results established in [3].

The filtering problem in the presence of FBM can be transformed (see [3]) into an equivalent deterministic optimal control problem, where its system dynamic is described by nonlinear ordinary differential equations involving convolutional integrals. It is very difficult to solve such an optimal control problem directly. The aim of this paper is to develop a computational scheme for solving this problem. First, as in [3], this filtering problem is transformed into a deterministic optimal control problem, where its system dynamic is described by nonlinear differential equations involving convolutional integrals. Then, a novel approximation scheme, supported by rigorous mathematical analysis, is developed to solve this optimal control problem.

We first introduce some background knowledge on FBM. For further details on FBM, see, for example, [5, 6, 15, 26], and for details on filtering problems, see [2, 29]. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $H \in (0, 1)$. $\mathbf{B}$ is an $n-$dimensional Brownian motion with covariance matrix $\mathbf{Q} \in \mathbf{M}_s^+(n \times n)$, where $\mathbf{M}_s^+(n \times n)$ denotes the class of all $n \times n$ real symmetric positive definite matrices. Define

$$\mathbf{B}_H(t) = \int_0^t K_H(t,\theta) d\mathbf{B}(\theta), \tag{1.1}$$

with K_H being a kernel depending on the parameter H. Let it be chosen as:

$$K_H(t,s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-\frac{t}{s}\right) \mathbf{1}_{(0,t)}(s), \tag{1.2}$$

where Γ is a gamma function and F is a hypergeometric function. $\mathbf{B}_H$ is a R^n-valued Gaussian random process with mean and covariance matrix given by

(i) $E\{\mathbf{B}_H(t)\} = 0$;
(ii) $E\{(\mathbf{B}_H(t), \xi)(\mathbf{B}_H(s), \eta)\} = \int_0^t \int_0^s \varphi_H(\tau - \theta)(\mathbf{Q}\xi, \eta) d\tau\, d\theta$ for all $\xi, \eta \in \mathbb{R}^n$.

From (ii), it follows that

$$E\{(\mathbf{B}_H(t), \xi)^2\} = t^{2H}(\mathbf{Q}\xi, \xi), \quad \xi \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{1.3}$$

where $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}$.

The rest of the paper is organized as follows. In Section 2, we formulate the filtering problem which is driven by a FBM process. As in [3], this filtering problem is shown to be equivalent to a deterministic optimal control problem with convolutional integrals appeared in its system dynamics. In Section 3, we construct a sequence of approximate optimal control problems, where their system dynamics are expressed by ordinary differential equations. The convergence properties of the approximation scheme are established in Section 4. Numerical simulation is presented in Section 5, while Section 6 concludes the paper.

2. LINEAR FILTERING WITH FBM

Consider the following FBM dynamical system:

$$d\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)dt + \mathbf{\Xi_1} d\mathbf{B}_{H_1}(t) \tag{2.1a}$$

$$\mathbf{x}(0) = \mathbf{x}_0, \tag{2.1b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, and $\{\mathbf{A}, \mathbf{\Xi_1}\}$ are $n \times n$ and $n \times d$ constant matrices, respectively. This model is known as the fractional Ornstein-Uhlenbeck process and it has a wide range of applications, see, for example, [7, 10, 12, 13].

The measurement dynamics is given by

$$d\mathbf{y}(t) = \mathbf{H}\mathbf{x}(t)dt + \mathbf{\Xi_2} d\mathbf{B}_{H_2}(t) \tag{2.2a}$$

$$\mathbf{y}(0) = \mathbf{0}, \tag{2.2b}$$

where $\mathbf{y}(t) \in \mathbb{R}^m$, and $\{\mathbf{H}, \mathbf{\Xi_2}\}$ are $m \times n$ and $m \times m$ constant matrices, respectively. $\{\mathbf{B}_{H_1}(t), t \geq 0\}$ and $\{\mathbf{B}_{H_2}(t), t \geq 0\}$ are FBM processes taking values in $\mathbb{R}^d$ and $\mathbb{R}^m$, respectively.

Let $\{\mathcal{F}_t^{\mathbf{y}}, t \geq 0\}$ be an increasing family of subsigma algebras of the sigma algebra $\mathcal{F}$ induced by the random process $\{\mathbf{y}(t), t \geq 0\}$. From [3], the basic filtering problem is to find a process $\mathbf{z}(t)$ so that for each $t \geq 0$, $\mathbf{z}(t)$ is $\mathcal{F}_t^{\mathbf{y}}$-adapted satisfying

$$(1) E\{\mathbf{z}(t)\} = E\{\mathbf{x}(t)\}, \quad t \geq 0; \text{ and} \tag{2.3a}$$

$$(2) E\{\| \mathbf{x}(t) - \mathbf{z}(t) \|^2\} \text{ is minimum for } t \geq 0, \tag{2.3b}$$

where $\| \cdot \|$ denotes the usual Euclidean norm. That is, for a vector $\mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{v}\| = \left(\sum_{i=1}^{n} (v_i)^2 \right)^{\frac{1}{2}}. \tag{2.4}$$

Furthermore, for a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$, we define

$$\|\mathbf{A}\| = (\sum_{i=1}^{n} \sum_{j=1}^{m} (A_{ij})^2)^{\frac{1}{2}} \tag{2.5}$$

and

$$\|\mathbf{A}\|_\infty = \max_{1 \leq i,j \leq n} |A_{i,j}|. \tag{2.6}$$

Such a $\mathbf{z}$ is known as the best unbiased-minimum variance (UMV) linear filter driven by the observation process $\mathbf{y}$. It is expressed in the form of the following stochastic differential equations

$$d\mathbf{z}(t) = \mathbf{G}_{\mathbf{\Gamma}}\mathbf{z}(t)dt + \mathbf{\Gamma} d\mathbf{y}(t), \quad t \geq 0, \tag{2.7a}$$

$$\mathbf{z}(0) = \hat{\mathbf{x}}_0 \equiv E\mathbf{x}_0, \tag{2.7b}$$

where $\mathbf{G}_{\mathbf{\Gamma}}$ and $\mathbf{\Gamma} \in \mathcal{D}$ are constant matrices with appropriate dimensions, which are to be determined, and

$$\mathcal{D} = \left\{ \mathbf{\Gamma} : \|\mathbf{\Gamma}\|_\infty = \max_{1 \leq i,j \leq n} |\Gamma_{i,j}| \leq M \right\}, \tag{2.8}$$

with $M > 0$ being a fixed constant. Let

$$\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{z}(t), \qquad t \geq 0. \tag{2.9}$$

Then,

$$\begin{aligned} d\mathbf{e} &= d\mathbf{x} - d\mathbf{z} \\ &= (\mathbf{A} - \mathbf{\Gamma}\mathbf{H})\mathbf{e}dt + (\mathbf{A} - \mathbf{\Gamma}\mathbf{H} - \mathbf{G}_{\mathbf{\Gamma}})\mathbf{z}(t)dt \\ &\quad + \mathbf{\Xi_1} d\mathbf{B}_{H_1}(t) - \mathbf{\Gamma}\mathbf{\Xi_2} d\mathbf{B}_{H_2}(t) \end{aligned} \tag{2.10a}$$

$$\mathbf{e}(0) = \mathbf{e}_0 \equiv \mathbf{x}_0 - \hat{\mathbf{x}}_0. \tag{2.10b}$$

Choose $\mathbf{G}_{\mathbf{\Gamma}} = \mathbf{A} - \mathbf{\Gamma}\mathbf{H}$. Then, it follows from (2.7) that

$$d\mathbf{z}(t) = (\mathbf{A} - \mathbf{\Gamma}\mathbf{H})\mathbf{z}(t)dt + \mathbf{\Gamma} d\mathbf{y}(t), \ t \geq 0, \tag{2.11a}$$

$$\mathbf{z}(0) = \hat{\mathbf{x}}_0. \tag{2.11b}$$

The error equation (2.10a) with initial condition (2.10b) is reduced to

$$d\mathbf{e} = (\mathbf{A} - \mathbf{\Gamma}\mathbf{H})\mathbf{e}(t)dt + \mathbf{\Xi_1} d\mathbf{B}_{H_1}(t) - \mathbf{\Gamma}\mathbf{\Xi_2} d\mathbf{B}_{H_2}(t), \tag{2.12a}$$

$$\mathbf{e}(0) = \mathbf{e}_0. \tag{2.12b}$$

Let $\{\mathbf{\Phi}(t,s), \ 0 \leq s \leq t < \infty\}$ denote the transition operator corresponding to $\mathbf{G}_{\mathbf{\Gamma}} = \mathbf{A} - \mathbf{\Gamma}\mathbf{H}$. With this operator, the solution of (2.12) can be written as:

$$\mathbf{e}(t) = \mathbf{\Phi}(t,0)\mathbf{e}_0 + \int_0^t \mathbf{\Phi}(t,\theta)\mathbf{\Xi_1} d\mathbf{B}_{H_1}(\theta) - \int_0^t \mathbf{\Phi}(t,\theta)\mathbf{\Gamma}\mathbf{\Xi_2} d\mathbf{B}_{H_2}(\theta). \tag{2.13}$$

The transition operator $\mathbf{\Phi}$ is governed by

$$\left(\frac{\partial}{\partial t}\right) \mathbf{\Phi}(t,s) = \mathbf{G}_{\mathbf{\Gamma}}\mathbf{\Phi}(t,s) \tag{2.14a}$$

and

$$\mathbf{\Phi}(t,t) = \mathbf{I}. \tag{2.14b}$$

Since $\mathbf{G}_{\mathbf{\Gamma}}$ is a constant matrix, we have

$$\mathbf{\Phi}(t,s) = e^{\mathbf{G}_{\mathbf{\Gamma}}(t-s)} = e^{\mathbf{G}_{\mathbf{\Gamma}}t}e^{-\mathbf{G}_{\mathbf{\Gamma}}s}. \tag{2.15}$$

We need Lemma 3.1 of [3], which is quoted below.

Lemma 1. For each $\mathbf{\Gamma} \in \mathcal{D}$, the error covariance matrix $\mathbf{K}$ is the solution of the system described by the following differential equations.

$$\begin{aligned}\dot{\mathbf{K}}(t) = \mathbf{G}_{\mathbf{\Gamma}}\mathbf{K}(t) + \mathbf{K}\mathbf{G}_{\mathbf{\Gamma}}^T &+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\left\{\int_0^t \varphi_{H_1}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\mathbf{\Xi_1}ds\right\}\mathbf{Q}\mathbf{\Xi_1}^T \\ &+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\left\{\int_0^t \varphi_{H_1}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\mathbf{\Xi_1}^T ds\right\}\mathbf{Q}^T\mathbf{\Xi_1} \\ &+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\left\{\int_0^t \varphi_{H_2}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\mathbf{\Gamma}\mathbf{\Xi}_2 ds\right\}\mathbf{Q}_0\mathbf{\Xi}_2^T\mathbf{\Gamma}^T \\ &+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\left\{\int_0^t \varphi_{H_2}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\mathbf{\Gamma}^T\mathbf{\Xi}_2^T ds\right\}\mathbf{Q}_0^T\mathbf{\Xi}_2\mathbf{\Gamma}\end{aligned} \tag{2.16a}$$

with initial condition

$$\mathbf{K}(0) = \mathbf{K}_0, \tag{2.16b}$$

where $\mathbf{Q} \in \mathbf{M}_s^+(d\times d)$ and $\mathbf{Q}_0 \in \mathbf{M}_s^+(m\times m)$ are covariance matrices of $\mathbf{B}_{H_1}(t)$ and $\mathbf{B}_{H_2}(t)$, respectively, and $\mathbf{G}_{\mathbf{\Gamma}} = \mathbf{A} - \mathbf{\Gamma}\mathbf{H}$.

Now, the filtering problem is transformed into the following equivalent optimal control problem.

Problem (P). Given the dynamic system

$$\begin{aligned}\dot{\mathbf{K}}(t) = \mathbf{G}_{\mathbf{\Gamma}}\mathbf{K}(t) + \mathbf{K}\mathbf{G}_{\mathbf{\Gamma}}^T &+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\left\{\int_0^t \varphi_{H_1}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\mathbf{\Xi_1}ds\right\}\mathbf{Q}\mathbf{\Xi_1}^T \\ &+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\left\{\int_0^t \varphi_{H_1}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\mathbf{\Xi_1}^T ds\right\}\mathbf{Q}^T\mathbf{\Xi_1} \\ &+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\left\{\int_0^t \varphi_{H_2}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\mathbf{\Gamma}\mathbf{\Xi_2} ds\right\}\mathbf{Q}_0\mathbf{\Xi_2}^T\mathbf{\Gamma}^T \\ &+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\left\{\int_0^t \varphi_{H_2}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\mathbf{\Gamma}^T\mathbf{\Xi_2}^T ds\right\}\mathbf{Q}_0^T\mathbf{\Xi_2}\mathbf{\Gamma},\end{aligned} \tag{2.17a}$$

$$\mathbf{K}(0) = \mathbf{K}_0, \tag{2.17b}$$

find a $\mathbf{\Gamma} \in \mathcal{D}$ such that the cost function

$$J(\mathbf{\Gamma}) = \int_0^T \mathrm{trace}\{\mathbf{\Sigma}(t)\mathbf{K}(t)\}dt \tag{2.18}$$

is minimized, where $\mathbf{\Sigma}(t)$ is a weighting matrix-valued function, which is positive definite and symmetric for each $t \in [0,T]$, and T is the terminal time.

This optimal control problem is very difficult to solve, because of the appearance of convolutional integrals in its system dynamics. In the next section, an approximation scheme is developed, which will then be used to approximate this optimal control problem into a sequence of optimal control problems involving only ordinary differential equations. Each of these standard approximate optimal control problems can be solved by the control parameterizations technique used in conjunction with the time scaling transform reported in [21, 31]. In particular, the optimal control software package, *MISER 3.3* [16], can be used for this purpose.

3. APPROXIMATION METHOD

In this section, we propose an approximation scheme to construct a sequence of approximate problems (*P(N)*) which are governed by ordinary differential equations. First, the kernels $\varphi_{H_1}(t-s)$ and $\varphi_{H_2}(t-s)$ in (2.16) are approximated by using an expansion of Chebyshev series. By doing this, the convolutional integrals are approximated by respective ordinary differential equations. In this way, Problem (P) is approximated by a sequence of optimal control problem involving only ordinary differential equations. Each of which can be solved by using many efficient numerical methods available in the literature.

We approximate the kernels $\varphi_{H_1}(t-s)$ and $\varphi_{H_2}(t-s)$ by a finite expansion of Chebyshev series as follows:

$$\varphi_{H_1}(t) \approx \varphi_{H_1}^N(t) = \sum_{i=1}^{N} \alpha_i^N k_i(t) \tag{3.1}$$

and

$$\varphi_{H_2}(t) \approx \varphi_{H_2}^N(t) = \sum_{i=1}^{N} \beta_i^N k_i(t) \tag{3.2}$$

where

(3.3)
$$k_i\,(t) = \bar{T}_{i-1}(t) = T_{i-1}\left(\frac{2t}{T}-1\right) = \cos\left[(i-1)\cos^{-1}\left(\frac{2t}{T}-1\right)\right], \quad 0 \le t \le T,$$

are basis functions obtained from the shifted Chebyshev series. They satisfy the system of ordinary differential equations with constant coefficients,

$$\dot{k}_i(t) = \sum_{j=1}^{N} a_{ij} k_j(t), \tag{3.4}$$

with initial condition

$$k_i(0) = \bar{T}_{i-1}(0) = T_{i-1}(-1) = (-1)^{i-1}, \tag{3.5}$$

where a_{ij}, α_i^N and β_i^N, $i = 1, \ldots, N$, $j = 1, \ldots, N$, are, given respectively, by

$$a_{ij} = \begin{cases} 0, & j \geq i \\ [2(i-1)/T][1-(-1)^{i-j}], & j \neq 1 \\ [(i-1)/T][1+(-1)^{i}], & j = 1 \end{cases} \tag{3.6}$$

$$\alpha_i^N = \begin{cases} \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \varphi_{H_1}(t_j), & i = 1 \\ \dfrac{2}{N}\displaystyle\sum_{j=1}^{N} \bar{T}_{i-1}(t_j)\varphi_{H_1}(t_j), & i = 2, \ldots, N \end{cases} \tag{3.7}$$

and

$$\beta_i^N = \begin{cases} \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \varphi_{H_2}(t_j), & i = 1 \\ \dfrac{2}{N}\displaystyle\sum_{j=1}^{N} \bar{T}_{i-1}(t_j)\varphi_{H_2}(t_j), & i = 2, \ldots, N, \end{cases} \tag{3.8}$$

while

$$t_j = \frac{T}{2} + \frac{T}{2}\cos\left[\frac{(2j-1)\pi}{2N}\right], \quad j = 1, \ldots, N. \tag{3.9}$$

With $\varphi_{H_1}(t)$ and $\varphi_{H_2}(t)$ being, respectively, approximated by $\varphi_{H_1}^N(t)$ and $\varphi_{H_2}^N(t)$ in (2.17), we obtain

$$\begin{aligned} \dot{\mathbf{K}}^N(t) = \mathbf{G_\Gamma K}^N(t) + \mathbf{K}^N\mathbf{G}_\mathbf{\Gamma}^T &+ e^{\mathbf{G_\Gamma}t}\left\{\int_0^t \varphi_{H_1}^N(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Xi_1}ds\right\}\mathbf{Q\Xi_1}^T \\ &+ e^{\mathbf{G_\Gamma}t}\left\{\int_0^t \varphi_{H_1}^N(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Xi_1}^T ds\right\}\mathbf{Q}^T\mathbf{\Xi_1} \\ &+ e^{\mathbf{G_\Gamma}t}\left\{\int_0^t \varphi_{H_2}^N(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Gamma\Xi_2}ds\right\}\mathbf{Q}_0\mathbf{\Xi_2}^T\mathbf{\Gamma}^T \\ &+ e^{\mathbf{G_\Gamma}t}\left\{\int_0^t \varphi_{H_2}^N(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Gamma}^T\mathbf{\Xi_2}^T ds\right\}\mathbf{Q}_0^T\mathbf{\Xi_2\Gamma} \end{aligned} \tag{3.10}$$

Now, Problem (P) is approximated by a sequence of optimal control problems (P(N)) defined below.

Problem (P(N)). Given the dynamical system (3.10) with initial condition

$$\boldsymbol{K}^N(0) = \boldsymbol{K}_0, \tag{3.11}$$

find a $\mathbf{\Gamma} \in \mathcal{D}$ such that the cost function

$$J(\mathbf{\Gamma}) = \int_0^T \mathrm{trace}\{\mathbf{\Sigma}(t)\mathbf{K}^N(t)\}dt \tag{3.12}$$

is minimized.

Define

$$\mathbf{w}_{1,i}(t) = \int_0^t k_i(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Xi_1}ds, \qquad i = 1, \ldots, N, \tag{3.13}$$

$$\mathbf{w}_{2,i}(t) = \int_0^t k_i(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Xi_1^{\boldsymbol{T}}}ds, \qquad i = 1, \ldots, N, \tag{3.14}$$

$$\mathbf{w}_{3,i}(t) = \int_0^t k_i(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Gamma}\mathbf{\Xi_2}ds, \qquad i = 1, \ldots, N. \tag{3.15}$$

and

$$\mathbf{w}_{4,i}(t) = \int_0^t k_i(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Gamma}^T\mathbf{\Xi_2}^Tds, \qquad i = 1, \ldots, N. \tag{3.16}$$

Taking the time derivative on both sides of (3.13), it follows from (3.4) that

$$\begin{aligned}\dot{\mathbf{w}}_{1,i}(t) &= k_i(0)e^{-\mathbf{G_\Gamma}t}\mathbf{\Xi_1} + \int_0^t \sum_{j=1}^N a_{ij}k_j(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Xi_1}ds \\ &= k_i(0)e^{-\mathbf{G_\Gamma}t}\mathbf{\Xi_1} + \sum_{j=1}^N a_{ij}\mathbf{w}_{1,j}(t),\end{aligned} \tag{3.17a}$$

with

$$\mathbf{w}_{1,i} = 0. \tag{3.17b}$$

Similarly, we have

$$\dot{\mathbf{w}}_{2,i}(t) = k_i(0)e^{-\mathbf{G_\Gamma}t}\mathbf{\Xi_1}^T + \sum_{j=1}^N a_{ij}\mathbf{w}_{2,j}(t), \tag{3.18a}$$

with

$$\mathbf{w}_{2,i} = 0, \tag{3.18b}$$

$$\dot{\mathbf{w}}_{3,i}(t) = k_i(0)e^{-\mathbf{G_\Gamma}t}\mathbf{\Gamma}\mathbf{\Xi_2} + \sum_{j=1}^N a_{ij}\mathbf{w}_{3,j}(t), \tag{3.19a}$$

with

$$\mathbf{w}_{3,i} = 0, \tag{3.19b}$$

and

$$\dot{\mathbf{w}}_{4,i}(t) = k_i(0)e^{-\mathbf{G_\Gamma}t}\mathbf{\Gamma}^T\mathbf{\Xi_2}^T + \sum_{j=1}^N a_{ij}\mathbf{w}_{4,j}(t), \tag{3.20a}$$

with

$$\mathbf{w}_{4,i} = 0. \tag{3.20b}$$

Since

$$\boldsymbol{\eta}_1(t) = \sum_{i=1}^{N} \alpha_i^N \int_0^t k_i(t-s) e^{-\mathbf{G}_{\mathbf{\Gamma}} s} \mathbf{\Xi_1} ds + \sum_{i=1}^{N} \alpha_i^N \int_0^t k_i(t-s) e^{-\mathbf{G}_{\mathbf{\Gamma}} s} \mathbf{\Xi_1}^T ds$$

$$= \sum_{i=1}^{N} \alpha_i^N (\mathbf{w}_{1,i}(t) + \mathbf{w}_{2,i}(t)), \tag{3.21}$$

and

$$\boldsymbol{\eta}_2(t) = \sum_{i=1}^{N} \beta_i^N \int_0^t k_i(t-s) e^{-\mathbf{G}_{\mathbf{\Gamma}} s} \mathbf{\Gamma} \mathbf{\Xi_2} ds + \sum_{i=1}^{N} \beta_i^N \int_0^t k_i(t-s) e^{-\mathbf{G}_{\mathbf{\Gamma}} s} \mathbf{\Gamma}^T \mathbf{\Xi_2}^T ds$$

$$= \sum_{i=1}^{N} \beta_i^N (\mathbf{w}_{3,i}(t) + \mathbf{w}_{4,i}(t)), \tag{3.22}$$

(3.10) is now approximated by the following system of ordinary differential equations

$$\begin{aligned}\dot{\mathbf{K}}^N = \mathbf{G}_{\mathbf{\Gamma}} \mathbf{K}^N(t) + \mathbf{K}^N \mathbf{G}_{\mathbf{\Gamma}}^T &+ e^{\mathbf{G}_{\mathbf{\Gamma}} t} \sum_{i=1}^{N} \alpha_i^N \mathbf{w}_{1,i}(t) \mathbf{Q} \mathbf{\Xi_1}^T \\ &+ e^{\mathbf{G}_{\mathbf{\Gamma}} t} \sum_{i=1}^{N} \alpha_i^N \mathbf{w}_{2,i}(t) \mathbf{Q}^T \mathbf{\Xi_1} \\ &+ e^{\mathbf{G}_{\mathbf{\Gamma}} t} \sum_{i=1}^{N} \beta_i^N \mathbf{w}_{3,i}(t) \mathbf{Q}_0 \mathbf{\Xi_2}^T \mathbf{\Gamma}^T \\ &+ e^{\mathbf{G}_{\mathbf{\Gamma}} t} \sum_{i=1}^{N} \beta_i^N \mathbf{w}_{4,i}(t) \mathbf{Q}_0^T \mathbf{\Xi_2} \mathbf{\Gamma},\end{aligned} \tag{3.23}$$

$$\dot{\mathbf{w}}_{1,i}(t) = k_i(0) e^{-\mathbf{G}_{\mathbf{\Gamma}} t} \mathbf{\Xi_1} + \sum_{j=1}^{N} a_{ij} \mathbf{w}_{1,j}(t), \tag{3.24}$$

$$\dot{\mathbf{w}}_{2,i}(t) = k_i(0) e^{-\mathbf{G}_{\mathbf{\Gamma}} t} \mathbf{\Xi_1}^T + \sum_{j=1}^{N} a_{ij} \mathbf{w}_{2,j}(t), \tag{3.25}$$

$$\dot{\mathbf{w}}_{3,i}(t) = k_i(0) e^{-\mathbf{G}_{\mathbf{\Gamma}} t} \mathbf{\Gamma} \mathbf{\Xi_2} + \sum_{j=1}^{N} a_{ij} \mathbf{w}_{3,j}(t), \tag{3.26}$$

$$\dot{\mathbf{w}}_{4,i}(t) = k_i(0) e^{-\mathbf{G}_{\mathbf{\Gamma}} t} \mathbf{\Gamma}^T \mathbf{\Xi_2}^T + \sum_{j=1}^{N} a_{ij} \mathbf{w}_{4,j}(t), \tag{3.27}$$

$$\mathbf{k}(0) = \mathbf{k}_0, \tag{3.28}$$

$$\mathbf{w}_{1,j}(0) = \mathbf{0}, \qquad \mathbf{w}_{2,j} = \mathbf{0}, \qquad \mathbf{w}_{3,j}(0) = \mathbf{0}, \qquad \mathbf{w}_{4,j} = \mathbf{0} \tag{3.29}$$

Let the optimal control problem with its system dynamics governed by (3.23)–(3.27) with initial conditions (3.28)–(3.29) be referred to as Problem $(\hat{P}(N))$. Problem (P(N)) is equivalent to Problem $(\hat{P}(N))$, as (3.10) with initial condition (3.11) is equivalent to (3.23)–(3.27) with initial conditions (3.28)-(3.29).

Remark 1. The system of differential equations (3.23)–(3.27) with initial conditions (3.28)–(3.29) is much easier to solve than the system of differential equations involving convolutional integrals given by (3.10) with initial condition (3.11).

4. ANALYSIS OF METHOD

In this section, we shall discuss some convergence properties relating to the approximation of Problem (P) by the sequence of approximate problems $(\hat{P}(N))$. We need Lemma 4.6 of [31], which is quoted below

Lemma 2. Let $R_1^N(t) = \varphi_{H_1}(t) - \varphi_{H_1}^N(t)$ and $R_2^N(t) = \varphi_{H_2}(t) - \varphi_{H_2}^N(t)$. Then,

$$\begin{aligned}\|R_1^N\|_\infty &\le \frac{C(N)(\frac{\pi T}{4})^m (N-m)!}{N!}\|\varphi_{H_1}^{(m)}\|_\infty \\ &= O(C(N)N^{-m})\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\|R_2^N\|_\infty &\le \frac{C(N)(\frac{\pi T}{4})^m (N-m)!}{N!}\|\varphi_{H_2}^{(m)}\|_\infty \\ &= O(C(N)N^{-m})\end{aligned} \tag{4.2}$$

where $m \le N-1$, $C(N) = \frac{2}{\pi}\log N + 2$ and $\|R_i^N\|_\infty = \max_{0\le t\le T}\|R_i^N(t)\|$ for $i = 1, 2$.

Theorem 1. *Let R_1^N and R_2^N be as defined in Lemma 2. Suppose that $\mathbf{K}(t)$ and $\mathbf{K}^N(t)$ are the error covariance matrices for Problem (P) and $(P(N))$, respectively. Then*

$$\lim_{N\to\infty}\|\mathbf{K}^N - \mathbf{K}\|_\infty = 0.$$

Proof. Define $\mathbf{u}^N(t) = \mathbf{K}(t) - \mathbf{K}^N(t)$. By taking the time derivative of $\mathbf{u}^N$, and using (2.16) and (3.10), we obtain

$$\begin{aligned}\dot{\mathbf{u}}^N(t) =& \mathbf{G_\Gamma u}^N(t) + \mathbf{u}^N(t)\mathbf{G_\Gamma}^T + e^{\mathbf{G_\Gamma}t}\{\int_0^t R_1^N(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Xi_1}ds\}\mathbf{Q\Xi_1}^T \\ &+ e^{\mathbf{G_\Gamma}t}\{\int_0^t R_1^N(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Xi_1}^T ds\}\mathbf{Q\Xi_1} \\ &+ e^{\mathbf{G_\Gamma}t}\{\int_0^t R_2^N(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Gamma\Xi_2}ds\}\mathbf{Q_0\Xi_2}^T\mathbf{\Gamma}^T \\ &+ e^{\mathbf{G_\Gamma}t}\{\int_0^t R_2^N(t-s)e^{-\mathbf{G_\Gamma}s}\mathbf{\Gamma}^T\mathbf{\Xi_2}^T ds\}\mathbf{Q}_0^T\mathbf{\Xi_2\Gamma}\end{aligned} \tag{4.3a}$$

with initial condition

$$\mathbf{u}^N(0) = \mathbf{0}. \tag{4.3b}$$

Let

$$
\begin{aligned}
\boldsymbol{\beta}^N(t) = {} & e^{\mathbf{G_\Gamma}t}\left\{\int_0^t R_1^N(t-s)e^{-\mathbf{G_\Gamma}s}\boldsymbol{\Xi_1}ds\right\}\mathbf{Q}\boldsymbol{\Xi_1}^T \\
& + e^{\mathbf{G_\Gamma}t}\left\{\int_0^t R_1^N(t-s)e^{-\mathbf{G_\Gamma}s}\boldsymbol{\Xi_1}^T ds\right\}\mathbf{Q}\boldsymbol{\Xi_1} \\
& + e^{\mathbf{G_\Gamma}t}\left\{\int_0^t R_2^N(t-s)e^{-\mathbf{G_\Gamma}s}\boldsymbol{\Gamma}\boldsymbol{\Xi_2}ds\right\}\mathbf{Q}_0\boldsymbol{\Xi_2}^T\boldsymbol{\Gamma}^T \\
& + e^{\mathbf{G_\Gamma}t}\left\{\int_0^t R_2^N(t-s)e^{-\mathbf{G_\Gamma}s}\boldsymbol{\Gamma}^T\boldsymbol{\Xi_2}^T ds\right\}\mathbf{Q}_0^T\boldsymbol{\Xi_2}\boldsymbol{\Gamma}
\end{aligned}
\tag{4.4}
$$

Then, we have

$$
\begin{aligned}
& \|\boldsymbol{\beta}^N(t)\| \\
& \leq \|e^{\mathbf{G_\Gamma}t}\|\left\{\int_0^t \|R_1^N(t-s)\|\|e^{-\mathbf{G_\Gamma}s}\|\|\boldsymbol{\Xi_1}\|ds\right\}\|\mathbf{Q}\|\|\boldsymbol{\Xi_1}^T\| \\
& + \|e^{\mathbf{G_\Gamma}t}\|\left\{\int_0^t \|R_1^N(t-s)\|\|e^{-\mathbf{G_\Gamma}s}\|\|\boldsymbol{\Xi_1}^T\|ds\right\}\|\mathbf{Q}\|\|\boldsymbol{\Xi_1}\| \\
& + \|e^{\mathbf{G_\Gamma}t}\|\left\{\int_0^t \|R_2^N(t-s)\|\|e^{-\mathbf{G_\Gamma}s}\|\|\boldsymbol{\Gamma}\|\|\boldsymbol{\Xi_2}\|ds\right\}\|\mathbf{Q}_0\|\|\boldsymbol{\Xi_2}^T\|\|\boldsymbol{\Gamma}^T\| \\
& + \|e^{\mathbf{G_\Gamma}t}\|\left\{\int_0^t \|R_2^N(t-s)\|\|e^{-\mathbf{G_\Gamma}s}\|\|\boldsymbol{\Gamma}^T\|\|\boldsymbol{\Xi_2}^T\|ds\right\}\|\mathbf{Q}_0^T\|\|\boldsymbol{\Xi_2}\|\|\boldsymbol{\Gamma}\|.
\end{aligned}
\tag{4.5}
$$

From Lemma 2, it follows that for any $\varepsilon > 0$, there exists an N_0 such that for all $N > N_0$, we have

$$\|R_1^N\|_\infty \leq \varepsilon \tag{4.6}$$

and

$$\|R_2^N\|_\infty \leq \varepsilon. \tag{4.7}$$

Furthermore, since $\mathbf{A}, \mathbf{H}, \boldsymbol{\Xi_1}$, and $\boldsymbol{\Xi_2}$ are all constant matrices and $e^{\mathbf{G_\Gamma}t}, e^{-\mathbf{G_\Gamma}t}$ and $\boldsymbol{\beta}^N(t)$ are all continuous on $[0, T]$, there exists a constant $\tilde{M}$ such that

$$\|\mathbf{A}\|_\infty = \max_{1\leq i,j\leq n}|A_{i,j}| \leq \tilde{M}, \qquad \|\mathbf{H}\|_\infty = \max_{1\leq i,j\leq n}|H_{i,j}| \leq \tilde{M},$$

$$\|e^{\mathbf{G_\Gamma}}\|_\infty = \max_{0\leq t\leq T}\|e^{\mathbf{G_\Gamma}t}\|_\infty \leq \tilde{M}, \qquad \|e^{-\mathbf{G_\Gamma}}\|_\infty = \max_{0\leq t\leq T}\|e^{-\mathbf{G_\Gamma}t}\|_\infty \leq \tilde{M},$$

$$\|\boldsymbol{\Xi_1}\|_\infty = \max_{1\leq i,j\leq n}|\Xi_{1(i,j)}| \leq \tilde{M}, \qquad \|\boldsymbol{\Xi_2}\|_\infty = \max_{1\leq i,j\leq n}|\Xi_{2(i,j)}| \leq \tilde{M}$$

$$\text{and} \qquad \|\boldsymbol{\beta}^N\|_\infty = \max_{0\leq t\leq T}\|\boldsymbol{\beta}^N(t)\| \leq \tilde{M},$$

where T is the terminal time of the process $x(t)$.

Thus, (4.5) is reduced to

$$
\begin{aligned}
\|\boldsymbol{\beta}^N(t)\| \leq\ & \|e^{\mathbf{G_\Gamma}t}\|\{\int_0^t \varepsilon\|e^{-\mathbf{G_\Gamma}s}\|\|\boldsymbol{\Xi_1}\|ds\}\|\mathbf{Q}\|\|\boldsymbol{\Xi_1}^T\| \\
& + \|e^{\mathbf{G_\Gamma}t}\|\{\int_0^t \varepsilon\|e^{-\mathbf{G_\Gamma}s}\|\|\boldsymbol{\Xi_1}^T\|ds\}\|\mathbf{Q}\|\|\boldsymbol{\Xi_1}\| \\
& + \|e^{\mathbf{G_\Gamma}t}\|\{\int_0^t \varepsilon\|e^{-\mathbf{G_\Gamma}s}\|\|\boldsymbol{\Gamma}\|\|\boldsymbol{\Xi_2}\|ds\}\|\mathbf{Q}_0\|\|\boldsymbol{\Xi_2}^T\|\|\boldsymbol{\Gamma}^T\| \\
& + \|e^{\mathbf{G_\Gamma}t}\|\{\int_0^t \varepsilon\|e^{-\mathbf{G_\Gamma}s}\|\|\boldsymbol{\Gamma}^T\|\|\boldsymbol{\Xi_2}^T\|ds\}\|\mathbf{Q}_0^T\|\|\boldsymbol{\Xi_2}\|\|\boldsymbol{\Gamma}\| \\
\leq\ & 4T\varepsilon\tilde{M} = \varepsilon\bar{M},
\end{aligned} \tag{4.8}
$$

for all $N > N_0$, where $\bar{M} = 4T\tilde{M}$. From (4.3) and (4.4), we have

$$
\begin{aligned}
&\dot{\mathbf{u}}^N(t) = \mathbf{G_\Gamma}\mathbf{u}^N(t) + \mathbf{u}^N(t)\mathbf{G}_{\mathbf{\Gamma}}^T + \boldsymbol{\beta}^N(t) \\
&\mathbf{u}^N(0) = \mathbf{0},
\end{aligned} \tag{4.9}
$$

with $\mathbf{G_\Gamma} = \mathbf{A} - \boldsymbol{\Gamma}\mathbf{H}$. Taking integration, we obtain

$$
\mathbf{u}^N(t) = \int_0^t \mathbf{G_\Gamma}\mathbf{u}^N(s)ds + \int_0^t \mathbf{u}^N(s)\mathbf{G}_{\mathbf{\Gamma}}^T ds + \int_0^t \boldsymbol{\beta}^N(s)ds. \tag{4.10}
$$

Note that

$$
\|\mathbf{G_\Gamma}\|_\infty = \max_{1\leq i,j\leq n} |G_{\mathbf{\Gamma}(i,j)}| \leq \|\mathbf{A}\|_\infty + M\|\mathbf{H}\|_\infty, \tag{4.11}
$$

where M is as defined in (2.8). Thus, for $N > N_0$,

$$
\begin{aligned}
\|\mathbf{u}^N(t)\| &= \|\int_0^t \mathbf{G_\Gamma}\mathbf{u}^N(s)ds + \int_0^t \mathbf{u}^N(s)\mathbf{G}_{\mathbf{\Gamma}}^T ds + \int_0^t \boldsymbol{\beta}^N(s)ds\| \\
&\leq \int_0^t \|\mathbf{G_\Gamma}\|\|\mathbf{u}^N(s)\|ds + \int_0^t \|\mathbf{u}^N(s)\|\|\mathbf{G}_{\mathbf{\Gamma}}^T\|ds + \int_0^t \|\boldsymbol{\beta}^N(s)\|ds. \\
&\leq 2(\|\mathbf{A}\|_\infty + M\|\mathbf{H}\|_\infty)\int_0^t \|\mathbf{u}^N(s)\|ds + \varepsilon\bar{M}.
\end{aligned} \tag{4.12}
$$

Therefore, by Gronwall's inequality, we have

$$
\|\mathbf{u}^N\|_\infty \leq \bar{M}\varepsilon \exp(|2(\|\mathbf{A}\|_\infty + M\|\mathbf{H}\|_\infty)|T)
$$

for $N > N_0$. Since $\varepsilon > 0$ is arbitrary, it follows that $\|\mathbf{u}^N\|_\infty \to \infty$ as $N \to \infty$. This completes the proof. □

From Theorem 1, we observe that Problem (*P(N)*) converges to the original problem (P) as $N \to \infty$. Thus, to solve Problem (P), we will solve a sequence of Problem (*P(N)*), where each of which involves only system of ordinary differential equations. There are several efficient optimization techniques that can be used. For example, the optimal control software, *MISER 3.3*, is applicable for this purpose.

5. NUMERICAL SIMULATION

In this section, we present some numerical examples to illustrate the applicability of our proposed method.

Example 5.1. Consider a system which is governed by the following stochastic differential equation:

$$dx = xdt + dB_{H_1}(t), \quad t \in [0,1] \tag{5.1a}$$

$$x(0) = 0, \tag{5.1b}$$

The measurement dynamics is given by

$$dy = 2xdt + dB_{H_2}(t), \quad t \in [0,1] \tag{5.2a}$$

$$y(0) = 0. \tag{5.2b}$$

Then, the filter system becomes

$$dz = (1-2r)\, z(t)\, dt + r dy(t), \quad t \geq 0, \tag{5.3a}$$

$$z(0) = \hat{x}_0 \equiv E\{x_0\} \tag{5.3b}$$

where $r \in [-10, 10]$ is the parameter to be determined.

Suppose that the statistics of this system are given as follows:

$$H_1 = H_2 = 0.8, \; Q = Q_0 = 0.01.$$

Then, r can be obtained by solving the following optimal control problem.

$$\min_r J(r) = \int_0^1 k(t)\, dt \tag{5.4}$$

subject to

$$\dot{k}(t) = 2(1-2r)\, k(t) + 0.04 \int_0^t \varphi_{H_1}(t-s)\, e^{(1-2r)(t-s)} ds \tag{5.5a}$$

$$k(0) = E\left\{(x_0 - \hat{x}_0)^2\right\}. \tag{5.5b}$$

Let this problem be referred to as Problem (Q). We construct the corresponding approximate optimal control Problem (Q(N))with $N = 7$, 13 and 19. The corresponding values of the optimal parameter r^* and the cost obtained are: $\{9.59496, 0.000428209\}$, $\{9.7218, 0.000433229\}$ and $\{9.9543, 0.000403976\}$, respectively. The time histories of the approximate states are plotted in Fig. 1. From Fig. 1, we see that the convergence is very fast with respect to N, and K^N with $N = 19$ can be regarded as the solution of system (5.5) with $r^* = 9.9543$.

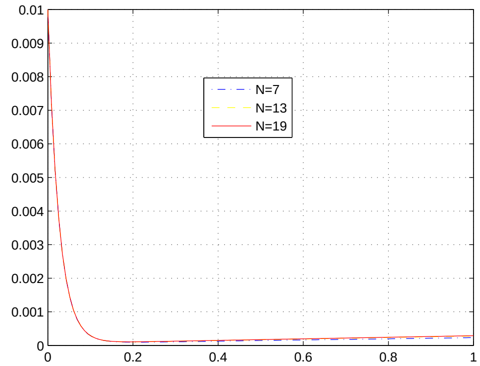

FIGURE 1. The state $k(t)$ with $N = 7$, $N = 13$ and $N = 19$.

Example 5.2. Consider a dynamical system given by the following stochastic differential equations defined on $[0, 1]$.

$$d\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} dt + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} d\mathbf{B}_{H_1}(t), \tag{5.6a}$$

$$\mathbf{x}(0) = [0\ 0]^T. \tag{5.6b}$$

The measurement system is given by

$$d\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} dt + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} d\mathbf{B}_{H_2}(t), \tag{5.7a}$$

$$\mathbf{y}(0) = [0\ 0]^T. \tag{5.7b}$$

Then, the filter system becomes

$$d\mathbf{z} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{z} dt + \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{bmatrix} d\mathbf{y}, \tag{5.8a}$$

$$\mathbf{z}(0) = \hat{\mathbf{x}}_0 = E\{\mathbf{x}_0\}. \tag{5.8b}$$

Suppose that $H_1 = H_2 = 0.8$, and that $\mathbf{Q} = \mathbf{Q}_0$ is an identity matrix. Then,

$$\tilde{\mathbf{Q}}(s, t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and } \tilde{\mathbf{Q}}_0(s, t) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{5.9}$$

Now we suppose that

$$\mathbf{\Gamma} = \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5.10}$$

where $\gamma_i \in [-10, 10]$, $i = 1, \ldots, 4$, are the parameters to be selected by solving the following optimal control problem.

$$\min_{\mathbf{\Gamma}} J(\mathbf{\Gamma}) = \int_0^1 \text{trace}\{\mathbf{\Sigma K}\} dt \tag{5.11}$$

subject to

$$
\begin{aligned}
\dot{\mathbf{K}}(t) = \mathbf{G}_{\mathbf{\Gamma}}(t)\mathbf{K}(t) + \mathbf{K}\mathbf{G}_{\mathbf{\Gamma}}^T(t) &+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\{\int_0^t \varphi_{H_1}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\tilde{\mathbf{Q}}(s,t)ds\} \\
&+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\{\int_0^t \varphi_{H_1}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\tilde{\mathbf{Q}}^T(s,t)ds\} \\
&+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\{\int_0^t \varphi_{H_2}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\mathbf{\Gamma}(s)\tilde{\mathbf{Q}}_0(s,t)ds\}\mathbf{\Gamma}^T(t) \\
&+ e^{\mathbf{G}_{\mathbf{\Gamma}}t}\{\int_0^t \varphi_{H_2}(t-s)e^{-\mathbf{G}_{\mathbf{\Gamma}}s}\mathbf{\Gamma}^T(s)\tilde{\mathbf{Q}}_0^T(s,t)ds\}\mathbf{\Gamma}(t),
\end{aligned}
\tag{5.12a}
$$

$$
\mathbf{K}(0) = \mathbf{K}_0, \tag{5.12b}
$$

where

$$
\mathbf{G}_{\mathbf{\Gamma}}(t) = \begin{bmatrix} -\gamma_1 & 1-\gamma_2 \\ 1-\gamma_3 & 1-\gamma_4 \end{bmatrix}, \; e^{\mathbf{G}_{\mathbf{\Gamma}}t} = \exp\{\int_0^t \begin{bmatrix} -\gamma_1 & 1-\gamma_2 \\ 1-\gamma_3 & 1-\gamma_4 \end{bmatrix}\}dt,
$$

$$
e^{-\mathbf{G}_{\mathbf{\Gamma}}t} = \exp\{-\int_0^t \begin{bmatrix} -\gamma_1 & 1-\gamma_2 \\ 1-\gamma_3 & 1-\gamma_4 \end{bmatrix}\}dt,
$$

$\tilde{\mathbf{Q}}$, $\tilde{\mathbf{Q}}_0$ and Γ from (5.9) and (5.10), respectively.

Let this problem be referred to as Problem (Q). We now construct the approximate optimal control Problem (*Q(N)*) with $N = 5$. Then, the optimal control software package, *MISER 3.3*, is used to solve such an approximate optimal control problem. The optimal Γ^* obtained is:

$$
\mathbf{\Gamma}^*(t) = \begin{bmatrix} 9.9965 & 1.12784 \\ 1.14684 & 9.9965 \end{bmatrix},
$$

and the optimal cost obtained is 0.104372705.

The time histories of the components of the state $\mathbf{K}^*$, i.e., the solution of the system (5.12) with $N = 5$ corresponding to $\mathbf{\Gamma} = \mathbf{\Gamma}^*$ are plotted in Fig. 2 and Fig. 3. For $N = 13$, the optimal cost obtained is 0.105569817 and the optimal $\mathbf{\Gamma}^*$ solution obtained is

$$
\mathbf{\Gamma}^* = \begin{bmatrix} 9.9987 & 1.1603 \\ 1.11603 & 9.9987 \end{bmatrix}.
$$

The time histories of the components of the corresponding $\mathbf{K}^*$ are plotted in Fig 4 and Fig 5.

From these examples, we can say that the method proposed is efficient. The figures show that the error covariance matrices converge very fast with respect to the observation data. These imply that if more observation data is available, then the estimation of $x(t)$ will be more accurate. The large error that can be seen occurs earlier

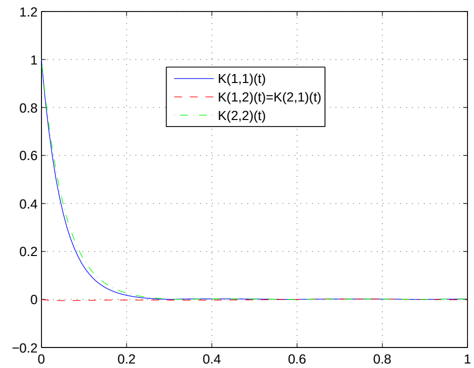

FIGURE 2. The state $K^*(t)$ with $N = 5$.

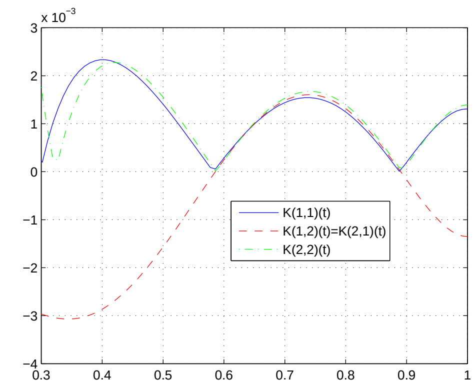

FIGURE 3. Zoom of the state $K^*(t)$ with $N = 5$.

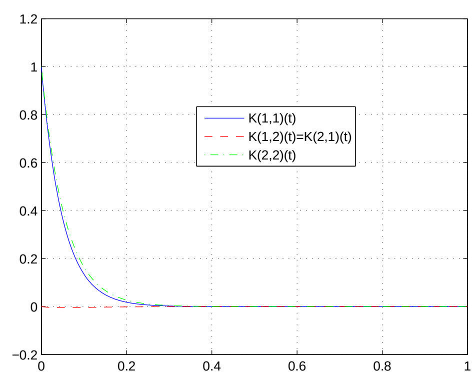

FIGURE 4. The state $K^*(t)$ with $N = 13$.

are basically caused by the fact that the initial error covariance matrices considered in these examples are large.

6. CONCLUSION

In this paper, we study the filtering problem in multi-dimensional linear system driven by a FBM process. We first showed that this filtering problem is equivalent to a deterministic optimal control problem involving convolutional integrals in its system

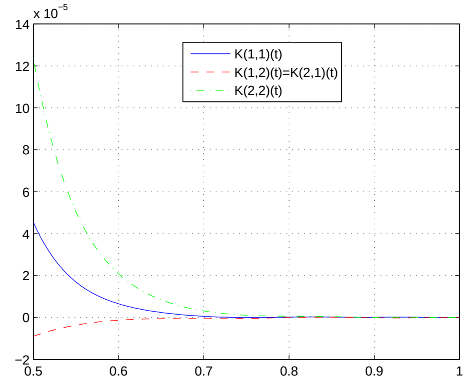

FIGURE 5. Zoom of the state $K^*(t)$ with $N = 13$.

dynamics. A computational scheme is developed, where we approximate the kernels appeared in the convolutional integrals by respective finite expansions of Chebyshev polynomials. With this approximation, the dynamical system that contains convolutional integrals is approximated by a sequence of systems involving only ordinary differential equations. The convergence of this approximation scheme was established. From the numerical simulation study, we observed that the method proposed is highly efficient.

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